Int. J. Solids Structures Vol. 29, No. 12, pp. 1499–1517, 1992
Printed in Great Britain.

0020–7683/92 $5.00+ .00

STRESS ANALYSIS OF ANISOTROPIC LAMINATED CYLINDERS AND CYLINDRICAL SEGMENTS

LÁSZLÓ P. KOLLÁR† and GEORGE S. SPRINGER
Department of Aeronautics and Astronautics, Stanford University, Stanford, CA 94305, U.S.A.

(*Received* 17 *May* 1991; *in revised form* 11 *October* 1991)

Abstract—A stress analysis of fiber-reinforced composite cylinders and cylindrical segments is presented. The analysis applies to thin as well as to thick walled cylinders with no restriction on fiber orientation, other than that an individual fiber must remain at the same radial distance from the axis. The cylinder may be subjected to hygrothermal and mechanical loads which may vary in the radial and circumferential, but not in the axial directions. Equations are derived which can be used to calculate the displacements, strains and stresses inside the material.

1. INTRODUCTION

Cylinders and cylindrical segments are important structural elements. For this reason many procedures have been put forth to analyse such elements made of isotropic materials. Relatively few analyses have been proposed pertaining to fiber-reinforced composite cylinders.

Shell approximations applicable to closed cylinders have been presented by many authors, and a detailed survey can be found in Noor *et al.* (1991). Analytical solutions taking into account three-dimensional variations in stresses and strains have been developed by Chou and Achenbach (1972), Noor and Rarig (1974), Srinivas (1974), Grigorenko *et al.* (1974), Chandrashekhara and Gopalakrishnan (1982), Hyer *et al.* (1986), Ren (1987), Hyer (1988), Roy and Tsai (1988), Noor and Peters (1989), Spencer *et al.* (1990), Varadan and Bhaskar (1991) and Lee and Springer (1990). All these investigators, except the last, analyse only orthotropic cylinders. Lee and Springer's analysis is for composite cylinders of arbitrary layup, but treats only radial stress distributions. No analysis seems to be available for generally anisotropic thick composite cylinders in which the stresses and strains vary both radially and circumferentially. Also, there appears to be no literature pertaining to the stress analysis of cylindrical segments.

Owing to the importance of the problem and to the lack of suitable analytical approaches, this investigation was undertaken to study the hygrothermal-mechanical behavior of composite cylinders. In particular, the objective was to develop analyses for calculating the behavior of fiber reinforced composite cylinders and cylindrical segments subjected to temperature, moisture and mechanical loads. In this paper the governing equations are described. Solutions applicable to closed cylinders are presented in a companion paper (Kollár *et al.*, 1992). Solutions for cylindrical segments and flat plates joined by rounded corners will be presented in subsequent publications.

An analytical approach was employed in this investigation instead of a finite element method. For large, thick structures finite element analysis may require excessive computer memory and computational time; in contrast, the method proposed here requires less computational effort.

2. PROBLEM STATEMENT

We consider a cylinder, or cylindrical segment (arc θ_0), made of n layers of unidirectional fiber reinforced composites (Figs 1 and 2). There is no restriction on either the

† On leave from the Technical University of Budapest, Department of Reinforced Concrete Structures, 1521 Hungary, Budapest.

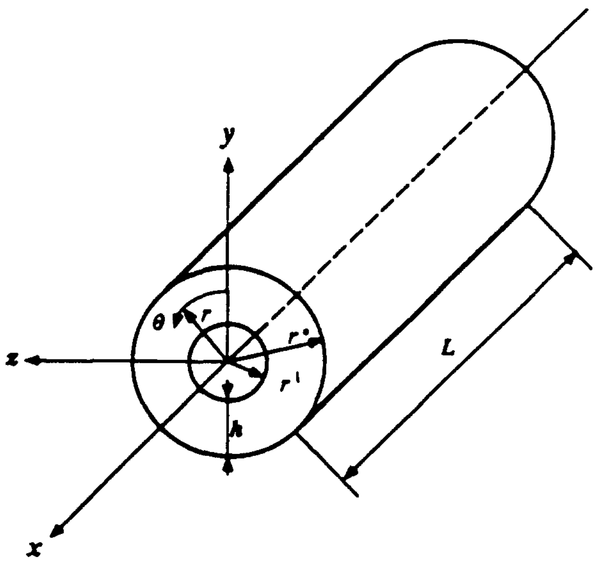

Fig. 1. Geometry of the closed cylinder.

number of plies or the orientation (ply-angle) of the fibers in each ply. Hence the cylinder may be "thick" and the layup may be unsymmetric. However, the cylinder must be long, so that the length L is large compared to the thickness h and to the inner r^i and outer r^o radii ($h/L \ll 1$, $r^o/L \ll 1$, $r^i/L \ll 1$). These approximations imply that edge effects are neglected.

The inner and outer surfaces of closed cylinders may be fixed or free (Fig. 3). The lengthwise edges of cylindrical segments may be fixed, simply supported, or free (Fig. 4).

Both the cylinder and the cylindrical segment may be subjected to hygrothermal and mechanical loads. These loads may vary in the radial r and circumferential θ directions,

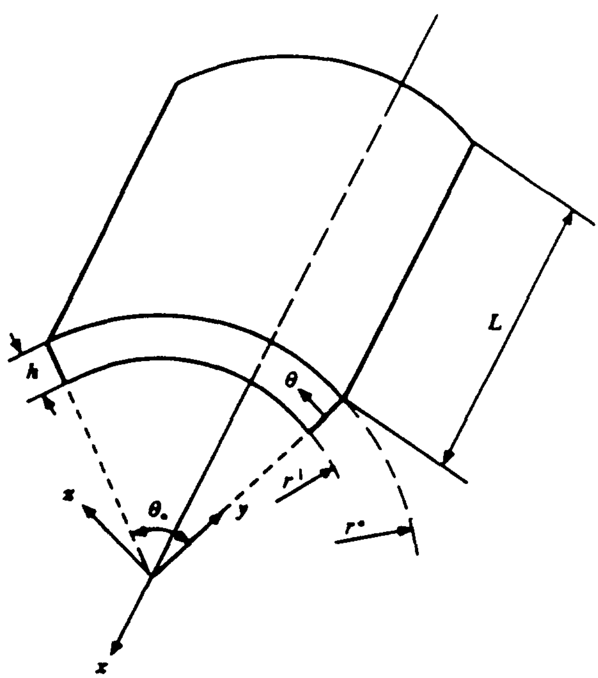

Fig. 2. Geometry of the cylindrical segment.

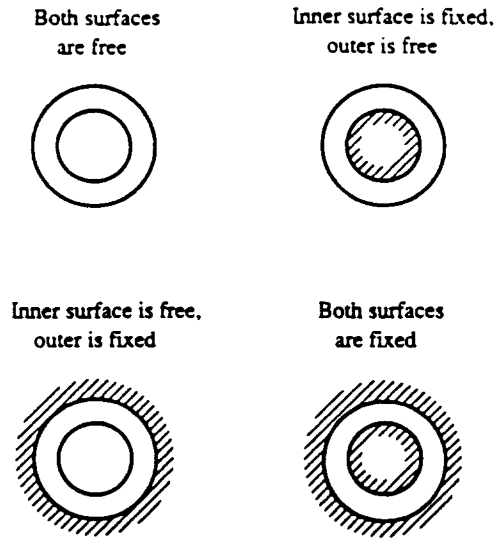

Fig. 3. The conditions on the inner and outer surfaces of the closed cylinder.

but must be independent of the axial coordinate x. Thus, the temperature ΔT and the moisture content Δc inside the composite may vary with r and θ but not with x. Here ΔT and Δc are known temperature and moisture content relative to prescribed reference values T_r and c_r

$$\Delta T(\theta, r) = T - T_r, \quad \Delta c(\theta, r) = c - c_r. \tag{1}$$

Mechanical loads may be imposed along the edges and on the surfaces as shown in Figs 5 and 6.

For a closed cylinder, loads can be imposed on the inner and outer surfaces in the radial, tangential, and axial directions. These loads, denoted by p_r^i, p_θ^i, p_x^i and p_r^o, p_θ^o, p_x^o, may vary with θ but not with x. Axial loads may also be placed along the edges of the closed cylinders and these loads are denoted by N_x. In addition, the cylinder may be subjected to a torque T and a bending moment M. The only restriction on the mechanical

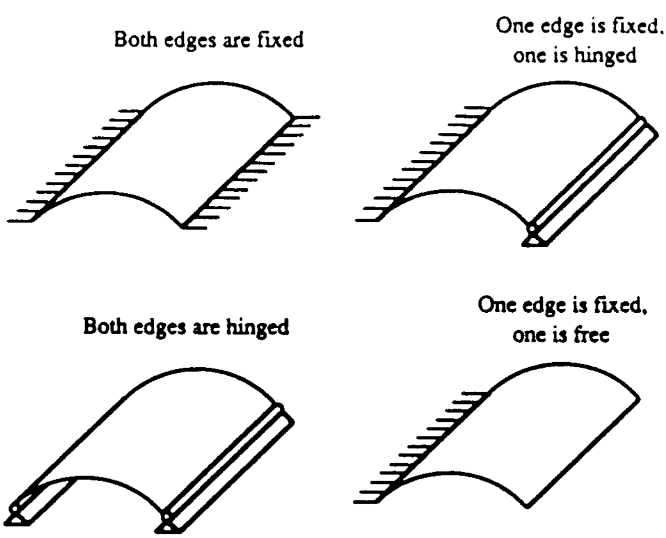

Fig. 4. The conditions along the lengthwise edges of the cylindrical segment.

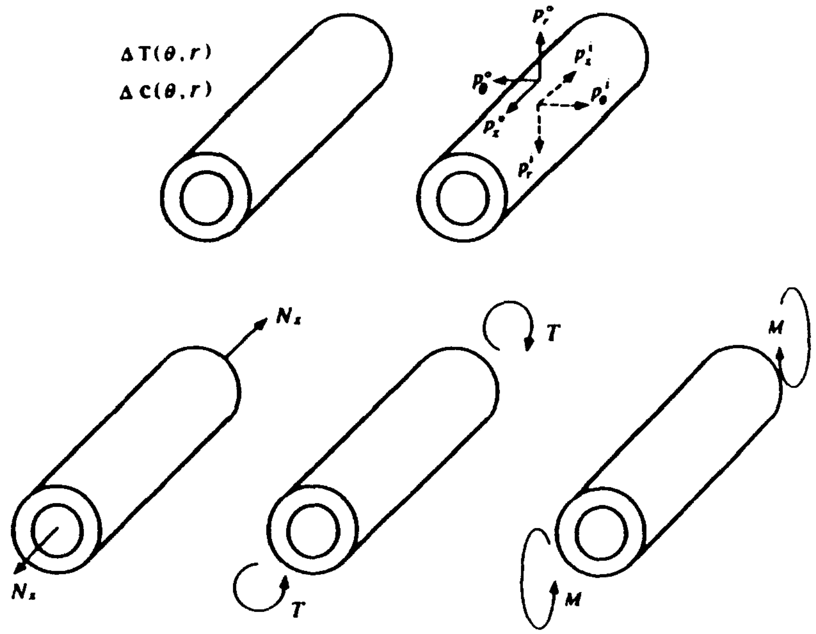

Fig. 5. The loads on the closed cylinder.

loads is that they must be in equilibrium, i.e. under their combined action the cylinder cannot undergo rigid body motion.

A cylindrical segment with edges unsupported may have edge loads on them as shown in Fig. 6. There may be axial Q_1 and shear loads Q_2 on the edges. The lengthwise, straight edges may also be subjected to loads Q_3 which act normal to the plane of symmetry, or to distributed moments Q_4. All these loads must be independent of x. In addition, the segment may be subjected to a torque T and a bending moment M. As in the case of closed cylinders, the only restriction on these loads is that the loads must be in equilibrium and must not result in rigid body motion.

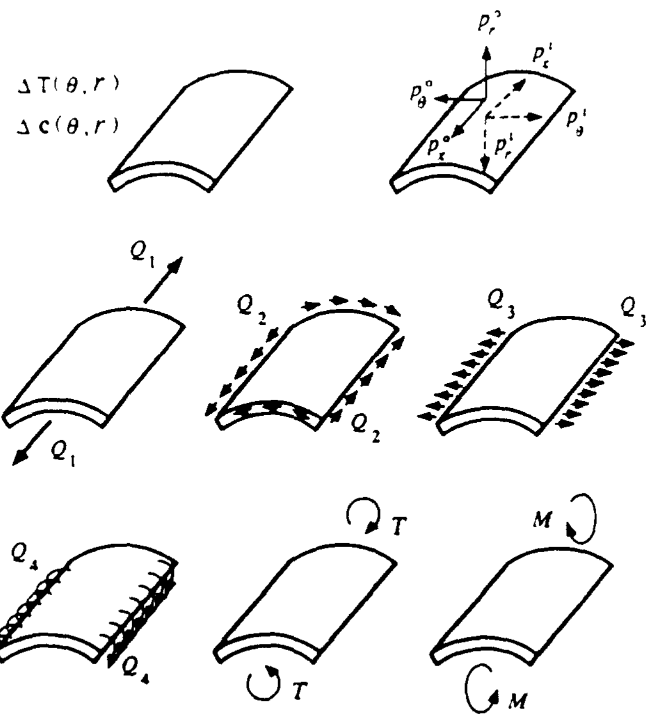

Fig. 6. The loads on the cylindrical segment.

A cylindrical segment supported along both of its longitudinal edges (simply supported or fixed) or built in along one edge and free along the other (Fig. 4) may be subjected to radial, circumferential, and axial loads on the inner and outer surfaces (Fig. 6). These loads may depend on θ but must be independent of x.

The objective is to find the stresses and strains inside the composite under the combined temperature, moisture and mechanical loads described above.

3. GOVERNING EQUATIONS

The analysis is applicable to loads which result in small deformations and linearly elastic material behavior. As described in the problem statement, all the loads, and hence all resulting strains and stresses are independent of the axial coordinate x. Then, the equations of equilibrium are (Love, 1944)

$$-\frac{\partial\tau_{xr}}{\partial r}-\frac{\tau_{xr}}{r}-\frac{\partial\tau_{x\theta}}{r\,\partial\theta}=0,\quad -\frac{\partial\sigma_\theta}{r\,\partial\theta}-\frac{\partial\tau_{\theta r}}{\partial r}-2\frac{\tau_{\theta r}}{r}=0,\quad +\frac{\sigma_\theta}{r}-\frac{\partial\sigma_r}{\partial r}-\frac{\sigma_r}{r}-\frac{\partial\tau_{\theta r}}{r\,\partial\theta}=0 \quad (2)$$

where, as usual, σ and τ represent normal and shear stresses. The strain displacement relations are (Love, 1944)

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_r=\frac{\partial w}{\partial r},\quad \varepsilon_\theta=\frac{w}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta},$$

$$\gamma_{\theta r}=\frac{\partial w}{r\,\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r},\quad \gamma_{xr}=\frac{\partial u}{\partial r}+\frac{\partial w}{\partial x},\quad \gamma_{x\theta}=\frac{\partial u}{r\,\partial\theta}+\frac{\partial v}{\partial x} \quad (3)$$

where ε is the normal strain and γ is the engineering shear strain. u, v and w are the displacements in the x, θ and r directions.

For the lth layer (ply), in the x, θ, r off-axis coordinate system the stress–strain relationship is (Tsai, 1988)

$$\begin{bmatrix}\sigma_x\\ \sigma_\theta\\ \sigma_r\\ \tau_{\theta r}\\ \tau_{xr}\\ \tau_{x\theta}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\ C_{21} & C_{22} & C_{23} & 0 & 0 & C_{26}\\ C_{31} & C_{32} & C_{33} & 0 & 0 & C_{36}\\ 0 & 0 & 0 & C_{44} & C_{45} & 0\\ 0 & 0 & 0 & C_{54} & C_{55} & 0\\ C_{61} & C_{62} & C_{63} & 0 & 0 & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x-\alpha_x\Delta T-\beta_x\Delta c\\ \varepsilon_\theta-\alpha_\theta\Delta T-\beta_\theta\Delta c\\ \varepsilon_r-\alpha_r\Delta T-\beta_r\Delta c\\ \gamma_{\theta r}\\ \gamma_{xr}\\ \gamma_{x\theta}-\alpha_{x\theta}\Delta T-\beta_{x\theta}\Delta c\end{bmatrix} \quad (4)$$

where C_{ij} ($=C_{ji}$, $i,j=1,\ldots,6$) are the components of the stiffness matrix; α and β are the thermal and moisture expansion coefficients.

Temperature and moisture

Temperature and moisture affect the strain in a similar manner. For simplicity, hereafter we include only ΔT in the analysis, with the understanding that the moisture content Δc can be included in an identical manner as the temperature.

The known temperature ΔT may be a function of r and θ, and may be expressed in the form of a Fourier series in the θ direction and in the form of a power series in the r direction

$$\Delta T=\sum_{j=0}\left\{\left[\sum_{i=0}\Delta T_{ji}(r)^i\right]\cos j\frac{\pi}{\theta_o}\theta+\left[\sum_{i=0}\Delta T^*_{ji}(r)^i\right]\sin j\frac{\pi}{\theta_o}\theta\right\} \quad (5)$$

where ΔT_{ji} and ΔT^*_{ji} are constants and are given by Kollár *et al.* (1992). When r is in parentheses, i is an exponent (not a superscript indicating inner radius).

Strains and displacements

The strains inside the composite may depend on r and θ. For analytical convenience we express the strain in three parts

$$\varepsilon(\theta, r) = \varepsilon_o(r) + \varepsilon_F(\theta, r) + \varepsilon_B(\theta, r). \tag{6}$$

ε_o is the function of r only. ε_F and ε_B depend on both θ and r. Furthermore, the solution is developed in such a way that in the case of ε_o and ε_F the axis of the cylinder remains straight, while in the case of ε_B the axis of the cylinder is curved. The displacements corresponding to each of these strains are identified by the subscripts o, F and B, and are

$$\begin{aligned} u^l(x, \theta, r) &= u^l_o(x, \theta, r) + u^l_F(\theta, r) + u^l_B(x, \theta, r) \\ v^l(x, \theta, r) &= v^l_o(x, \theta, r) + v^l_F(\theta, r) + v^l_B(x, \theta, r) \\ w^l(x, \theta, r) &= w^l_o(x, \theta, r) + w^l_F(\theta, r) + w^l_B(x, \theta, r). \end{aligned} \tag{7}$$

The superscript l refers to the lth layer. To simplify the notation, we omit this superscript when dealing with one layer in Sections 4–7. We retain the superscript l when analysing multilayer laminates in Section 9.

Our task is now to derive the appropriate relationships for the nine displacement terms on the right-hand side of eqn (7).

4. RADIALLY VARYING STRAINS AND STRESSES

First we analyse a problem in which the strains, and consequently the stresses, vary in the r direction only. In this case the temperature within the composite must vary with r only. This condition is met when $j = 0$ in eqn (5). Thus the expression for ΔT becomes

$$\Delta T = \sum_{i=0} \Delta T_{oi}(r)^i \tag{8}$$

where ΔT_{oi} is defined in Kollár *et al.* (1992).

The cylinder or cylindrical segment may undergo three translations and three rotations about the x, y, z axes. In the problems considered here all these rigid body motions are absent. However, for the purpose of the analysis we retain two of these motions: the displacement along and the rotation about the x axis. Then, the most general form of the displacement field, which satisfies the condition that the strain is a function of r only, is

$$u_o = u_a x + u_b \theta + u_c(r), \quad v_o = v_a x r + v_b \theta r + v_c(r), \quad w_o = w_o(r) \tag{9}$$

where u_a, u_b, v_a, v_b are constants. By using these displacements, the strains [eqn (3)] and the stresses [eqn (4)] are evaluated, and the resulting stresses are substituted into eqns (2). This procedure yields

$$\begin{aligned} &C_{55}\left(\frac{\partial^2 u_c}{\partial r^2} + \frac{\partial u_c}{r\,\partial r}\right) + C_{45}\left(\frac{\partial^2 v_c}{\partial r^2}\right) = 0 \\ &C_{45}\left(\frac{\partial^2 u_c}{\partial r^2} + 2\frac{\partial u_c}{r\,\partial r}\right) + C_{44}\left(\frac{\partial^2 v_c}{\partial r^2} + \frac{\partial v_c}{r\,\partial r} - \frac{v_c}{r^2}\right) = 0 \\ &C_{33}\left(\frac{\partial^2 w_o}{\partial r^2} + \frac{\partial w_o}{r\,\partial r}\right) - C_{22}\frac{w_o}{r^2} - \frac{\delta_1}{r^2} - \frac{\delta_2}{r} - \delta_3 - \sum_{i=0} \Delta_i (r)^{i-1} = 0. \end{aligned} \tag{10}$$

Table 1. Definition of the parameters in eqns (10) and (13)

$$\delta_1 = u_b C_{26}$$

$$\delta_2 = u_a(C_{21} - C_{31}) + v_b(C_{22} - C_{32})$$

$$\delta_3 = v_a(C_{26} - 2C_{36})$$

$$\Delta_i = -\Delta T_{oi}[\alpha_x(C_{21} - (1+i)C_{31}) + \alpha_\theta(C_{22} - (1+i)C_{32}) + \alpha_r(C_{23} - (1+i)C_{33}) + \alpha_{x\theta}(C_{26} - (1+i)C_{36})]$$

$$(i = 0, 1, \ldots)$$

$$\lambda = \sqrt{\frac{C_{22}}{C_{33}}}$$

$$f_i(r) = \frac{1}{(i+1)^2 C_{33} - C_{22}}\left(\frac{r}{R}\right)^{i+1} \quad \text{if} \quad (i+1)^2 C_{33} - C_{22} \neq 0$$

$$f_i(r) = \frac{\ln \frac{r}{R}}{2(i+1)C_{33}}\left(\frac{r}{R}\right)^{i+1} \quad \text{if} \quad (i+1)^2 C_{33} - C_{22} = 0$$

The first two of the above equations result in the following expressions for u_c, v_c

$$u_c(r) = u_c \ln\left(\frac{r}{R}\right) + u_d - 2v_c \frac{C_{45}}{C_{55}} \frac{R}{r} \tag{11}$$

$$v_c(r) = v_c \frac{1}{r} + v_d \frac{r}{R} + u_c \frac{C_{45}}{C_{44}}. \tag{12}$$

Equation (10c) yields

$$w_o(r) = A_1\left(\frac{r}{R}\right)^{\lambda} + A_2\left(\frac{r}{R}\right)^{-\lambda} - \frac{\delta_1}{C_{22}} + \delta_2 R f_o(r) + \delta_3 R^2 f_1(r) + \sum_{i=0} \Delta_i R^{i+1} f_i(r). \tag{13}$$

δ_1, δ_2, δ_3, Δ_i, λ and $f_i(r)$ are parameters defined in Table 1. R is a reference radius. A suitable choice for R is the radius of the mid-surface. u_a, u_b, u_c, u_d, v_a, v_b, v_c, v_d, A_1, A_2 are as yet undetermined constants. Thus, there are a total of 10 unknown constants for each ply.

Inspection of eqns (11) and (12) shows that u_d is the rigid body displacement in the axial direction, and $v_d(r/R)$ represents angular displacements about the x-axis.

The strains and stresses calculated from the above displacements u_o, v_o, w_o are identified by the subscript o, i.e. the resulting strain components are: ε_{xo}, $\varepsilon_{\theta o}$, ε_{ro}, $\gamma_{\theta ro}$, γ_{xro}, $\gamma_{x\theta o}$, and the resulting stress components are: σ_{xo}, $\sigma_{\theta o}$, σ_{ro}, $\tau_{\theta ro}$, τ_{xro}, $\tau_{x\theta o}$.

5. RADIALLY AND CIRCUMFERENTIALLY VARYING STRAINS AND STRESSES (STRAIGHT AXIS)

Next we consider a cylinder, or a cylindrical segment, in which the strains and the stresses may vary with r and θ, but where the axis of the cylinder remains straight. The radii of curvature of the axis are related to the displacements through the expressions

$$\kappa^y = -\frac{\partial^2 w(x, \theta, r)}{\partial x^2}, \text{ at } \theta = 0; \quad \kappa^z = -\frac{\partial^2 w(x, \theta, r)}{\partial x^2}, \text{ at } \theta = \frac{\pi}{2}. \tag{14}$$

κ^y and κ^z are the radii of curvature in the x–y and x–z planes.

The following form of displacements satisfies the requirement that the axis remains straight

$$u_F(\theta, r) = \sum_{j=1} \left\{ u_j(r) \sin j \frac{\pi}{\theta_o} \theta \right\} - \sum_{j=1} \left\{ u_j^*(r) \cos j \frac{\pi}{\theta_o} \theta \right\}$$

$$v_F(\theta, r) = \sum_{j=1} \left\{ v_j(r) \sin j \frac{\pi}{\theta_o} \theta \right\} - \sum_{j=1} \left\{ v_j^*(r) \cos j \frac{\pi}{\theta_o} \theta \right\}$$

$$w_F(\theta, r) = \sum_{j=1} \left\{ w_j(r) \cos j \frac{\pi}{\theta_o} \theta \right\} + \sum_{j=1} \left\{ w_j^*(r) \sin j \frac{\pi}{\theta_o} \theta \right\}. \tag{15}$$

Note that the summation of the series starts at $j = 1$. The oth term was discussed in the previous section. Correspondingly, the temperature [eqn (5)] is also only evaluated for $j \geq 1$, i.e.

$$\Delta T = \sum_{j=1} \left\{ \left[\sum_{i=0} \Delta T_{ji}(r)^i \right] \cos j \frac{\pi}{\theta_o} \theta + \left[\sum_{i=0} \Delta T_{ji}^*(r)^i \right] \sin j \frac{\pi}{\theta_o} \theta \right\}. \tag{16}$$

In the following we derive expressions for u_j, v_j, w_j and u_j^*, v_j^*, w_j^*. For simplicity we only show the derivation for one of the terms in each of the displacements in eqn (15) and in the temperature [eqn (16)]. The terms to be discussed in detail are ($j \geq 1$)

$$u_j(r) \sin j \frac{\pi}{\theta_o} \theta, \quad v_j(r) \sin j \frac{\pi}{\theta_o} \theta, \quad w_j(r) \cos j \frac{\pi}{\theta_o} \theta, \quad \left[\sum_{i=0} \Delta T_{ji}(r)^i \right] \cos j \frac{\pi}{\theta_o} \theta. \tag{17}$$

Subsequently, the results will be generalized to include every term of the series.

The strains [eqn (3)] are calculated with the displacements given by eqn (17). The stresses [eqn (4)] are then evaluated with these strains together with the temperature given in eqn (17). Substitution of the resulting stresses into the equilibrium equations [eqn (2)] yields

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} \begin{bmatrix} u_j(r) \\ v_j(r) \\ w_j(r) \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \tag{18}$$

where Ω_{ij} and the parameters B_1, B_2, B_3 are defined in Table 2. Equation (18) is a sixth order ordinary equidimensional differential equation system. Solution of these equations yields $u_j(r)$, $v_j(r)$, $w_j(r)$.

Solution of the homogeneous equation

When the temperature difference is zero ($\Delta T = 0$), B_1, B_2 and B_3 are zero (Table 2), and eqn (18) reduces to

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} \begin{bmatrix} u_j(r) \\ v_j(r) \\ w_j(r) \end{bmatrix} = 0. \tag{19}$$

For this homogeneous equidimensional differential equation system, we seek a solution of the form

$$u_j(r) = G_j^u \left(\frac{r}{R} \right)^{\gamma}, \quad v_j(r) = G_j^v \left(\frac{r}{R} \right)^{\gamma}, \quad w_j(r) = G_j^w \left(\frac{r}{R} \right)^{\gamma} \tag{20}$$

Table 2. Definition of the symbols in eqn (18)

$$\Omega_{11} = -C_{55}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right] + C_{66}j^2\left(\frac{\pi}{\theta_o}\right)^2\frac{1}{r^2}$$

$$\Omega_{12} = -C_{45}\frac{\partial^2}{\partial r^2} - C_{26}j^2\left(\frac{\pi}{\theta_o}\right)^2\frac{1}{r^2}$$

$$\Omega_{13} = (C_{45}+C_{36})\frac{1}{r}\frac{\partial}{\partial r}j\frac{\pi}{\theta_o} - C_{26}j\frac{\pi}{\theta_o}\frac{1}{r^2}$$

$$\Omega_{21} = -C_{45}\left[\frac{\partial^2}{\partial r^2} - 2\frac{1}{r}\frac{\partial}{\partial r}\right] + C_{26}j^2\left(\frac{\pi}{\theta_o}\right)^2\frac{1}{r^2}$$

$$\Omega_{22} = -C_{44}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right] + C_{22}j^2\left(\frac{\pi}{\theta_o}\right)^2\frac{1}{r^2}$$

$$\Omega_{23} = \frac{1}{r}\frac{\partial}{\partial r}j\frac{\pi}{\theta_o}(C_{44}+C_{23}) + \frac{1}{r^2}j\frac{\pi}{\theta_o}(C_{44}+C_{22})$$

$$\Omega_{31} = -C_{36}\frac{1}{r}\frac{\partial}{\partial r}j\frac{\pi}{\theta_o} + C_{26}j\frac{\pi}{\theta_o}\frac{1}{r^2}$$

$$\Omega_{32} = -\frac{1}{r}\frac{\partial}{\partial r}j\frac{\pi}{\theta_o}(C_{44}+C_{23}) + \frac{1}{r^2}j\frac{\pi}{\theta_o}(C_{44}+C_{22})$$

$$\Omega_{33} = -C_{33}\frac{\partial^2}{\partial r^2} - C_{33}\frac{1}{r}\frac{\partial}{\partial r} + C_{22}\frac{1}{r^2} + C_{44}j^2\left(\frac{\pi}{\theta_o}\right)^2\frac{1}{r^2}$$

$$B_1 = \left[\sum_{i=0}\Delta T_i r^{i-1}\right]q_1 j\frac{\pi}{\theta_o}$$

$$B_2 = \left[\sum_{i=0}\Delta T_i r^{i-1}\right]q_2 j\frac{\pi}{\theta_o}$$

$$B_3 = \sum_{i=0}[\Delta T_i r^{i-1}(q_2-(i+1)q_3)]$$

$$q_1 = C_{61}\alpha_x + C_{62}\alpha_\theta + C_{63}\alpha_r + C_{66}\alpha_{x\theta}$$
$$q_2 = C_{21}\alpha_x + C_{22}\alpha_\theta + C_{23}\alpha_r + C_{26}\alpha_{x\theta}$$
$$q_3 = C_{31}\alpha_x + C_{32}\alpha_\theta + C_{33}\alpha_r + C_{36}\alpha_{x\theta}$$

where γ, G_i^u, G_i^v, G_i^w are constants. By substituting eqn (20) into eqn (19), and after algebraic manipulations, we obtain

$$E(\gamma)\begin{bmatrix} G_i^u \\ G_i^v \\ G_i^w \end{bmatrix} = 0 \tag{21}$$

where the matrix E depends on the exponent γ and is defined in Table 3.

For a non-trivial solution of eqn (21), the determinant of the E matrix must be zero

$$\det(\mathbf{E}) = 0. \tag{22}$$

The determinant of **E** is a sixth order polynomial in γ. Since there are only even powers of γ, the polynomial can be reduced to a third order one, and this greatly simplifies the solution.

The case when $j(\pi/\theta_o) \neq 1$. When $j(\pi/\theta_o) \neq 1$, eqn (22) provides six independent solutions for γ, and these we denote as $\gamma_1, \gamma_2, \ldots, \gamma_6$. Note that $\gamma_1, \gamma_2, \ldots, \gamma_6$ can be real or complex numbers. The displacements [see eqn (20)] corresponding to each of the six γ

Table 3. The matrix **E** in eqn (21)

$$\mathbf{E} = \begin{bmatrix} C_{66}j^2\left(\frac{\pi}{\theta_o}\right)^2 - C_{55}\gamma^2 & C_{26}j^2\left(\frac{\pi}{\theta_o}\right)^2 - C_{45}(-\gamma+\gamma^2) & j\frac{\pi}{\theta_o}(C_{26}+\gamma(C_{36}-C_{45})) \\ C_{26}j^2\left(\frac{\pi}{\theta_o}\right)^2 - C_{45}(\gamma+\gamma^2) & C_{44}+C_{22}j^2\left(\frac{\pi}{\theta_o}\right)^2 - C_{44}\gamma^2 & j\frac{\pi}{\theta_o}(C_{22}+C_{44}+\gamma(C_{23}+C_{44})) \\ j\frac{\pi}{\theta_o}(C_{26}-\gamma(C_{36}+C_{45})) & j\frac{\pi}{\theta_o}(C_{22}+C_{44}-\gamma(C_{23}+C_{44})) & C_{22}+C_{44}j^2\left(\frac{\pi}{\theta_o}\right)^2 - C_{33}\gamma^2 \end{bmatrix}$$

values can be expressed as

$$u_{jk}^{\text{hom}}(r) = G_{jk}^{u}\left(\frac{r}{R}\right)^{\gamma_k}, \quad v_{jk}^{\text{hom}}(r) = G_{jk}^{v}\left(\frac{r}{R}\right)^{\gamma_k}, \quad w_{jk}^{\text{hom}}(r) = G_{jk}^{w}\left(\frac{r}{R}\right)^{\gamma_k} \tag{23}$$

where $k = 1, 2, \ldots, 6$.

The solution of the homogeneous equation [eqn (19)] is then the sum of the six displacements

$$u_j^{\text{hom}}(r) = \sum_{k=1}^{6} u_{jk}^{\text{hom}}(r), \quad v_j^{\text{hom}}(r) = \sum_{k=1}^{6} v_{jk}^{\text{hom}}(r), \quad w_j^{\text{hom}}(r) = \sum_{k=1}^{6} w_{jk}^{\text{hom}}(r). \tag{24}$$

These equations contain 18 constants G_{jk}^{u}, G_{jk}^{v}, G_{jk}^{w} $(k = 1, 2, \ldots, 6)$. The vectors $[G_{jk}^{u}, G_{jk}^{v}, G_{jk}^{w}]^{\mathrm{T}}$ must satisfy eqn (21). Hence only six of these 18 G-values are independent. We may select either G_{jk}^{u}, G_{jk}^{v} or G_{jk}^{w} as an independent variable and denote the one selected by the symbol G_{jk}. Thus we write

$$G_{jk}^{u} \equiv G_{jk} \quad \text{or} \quad G_{jk}^{v} \equiv G_{jk} \quad \text{or} \quad G_{jk}^{w} \equiv G_{jk}. \tag{25}$$

By substituting the first, second or third of eqn (25) into eqn (21) we obtain

$$\begin{bmatrix} G_{jk}^{v} \\ G_{jk}^{w} \end{bmatrix} = -G_{jk}\begin{bmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{bmatrix}^{-1}\begin{bmatrix} E_{21} \\ E_{31} \end{bmatrix}, \quad (G_{jk}^{u} \equiv G_{jk})$$

or

$$\begin{bmatrix} G_{jk}^{u} \\ G_{jk}^{w} \end{bmatrix} = -G_{jk}\begin{bmatrix} E_{11} & E_{13} \\ E_{31} & E_{33} \end{bmatrix}^{-1}\begin{bmatrix} E_{12} \\ E_{32} \end{bmatrix}, \quad (G_{jk}^{v} \equiv G_{jk})$$

or

$$\begin{bmatrix} G_{jk}^{u} \\ G_{jk}^{v} \end{bmatrix} = -G_{jk}\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}^{-1}\begin{bmatrix} E_{13} \\ E_{23} \end{bmatrix}, \quad (G_{jk}^{w} \equiv G_{jk}) \tag{26}$$

where E_{11}, E_{12}, etc. are the elements of the matrix **E** (see Table 3) with γ replaced by γ_k. We select the one of the above three equations for which the coefficient matrix is non-singular. Once the six unknowns G_{jk} $(k = 1, 2, \ldots, 6)$ are known, G_{jk}^{u}, G_{jk}^{v} and G_{jk}^{w} can be evaluated from the applicable expression in eqn (26).

The case when $j(\pi/\theta_o) = 1$. The above solution is inapplicable when $j(\pi/\theta_o) = 1$. In this case the determinant of the matrix **E** has only five independent roots. It can be shown, e.g. by the use of a symbolic manipulator, such as "Mathematica" (Wolfram, 1988), that two of the γ roots have the same value and are equal to zero. We arbitrarily select γ_5 and

γ_6 as the identical terms

$$\gamma_5 = \gamma_6 = 0. \tag{27}$$

Now, the expressions for the displacements in eqn (23) become

$$u_{j5}^{\text{hom}}(r) = G_{j5}^{u}, \quad v_{j5}^{\text{hom}}(r) = G_{j5}^{v}, \quad w_{j5}^{\text{hom}}(r) = G_{j5}^{w} \tag{28}$$

$$u_{j6}^{\text{hom}}(r) = G_{j6}^{u}, \quad v_{j6}^{\text{hom}}(r) = G_{j6}^{v}, \quad w_{j6}^{\text{hom}}(r) = G_{j6}^{w}. \tag{29}$$

We evaluate G_{j5}^{u}, G_{j5}^{v} and G_{j6}^{u}, G_{j6}^{v} from eqn (26). The components of $\mathbf{E}$ (E_{11}, E_{12}, etc.) are calculated by setting (in Table 3) $\gamma = \gamma_5 = 0$ and $\gamma = \gamma_6 = 0$. The result is

$$G_{j5}^{u} = 0, \quad G_{j5}^{v} = -G_{j5}, \quad G_{j5}^{w} = G_{j5}, \quad G_{j6}^{u} = 0, \quad G_{j6}^{v} = -G_{j6}, \quad G_{j6}^{w} = G_{j6}. \tag{30}$$

With reference to eqn (15) it can be shown that G_{j5} and G_{j6} represent rigid body displacements in the $\theta = 0$ direction.

Since $\gamma_5 = \gamma_6$, the displacements given by eqns (28) and (29) are identical. We now seek an independent sixth solution of the form

$$\bar{u}_{j6}^{\text{hom}}(r) = \bar{K}_{j6}^{u} + \bar{L}_{j6}^{u} \ln\frac{r}{R}, \quad \bar{v}_{j6}^{\text{hom}}(r) = \bar{K}_{j6}^{v} + \bar{L}_{j6}^{v} \ln\frac{r}{R}, \quad \bar{w}_{j6}^{\text{hom}}(r) = \bar{K}_{j6}^{w} + \bar{L}_{j6}^{w} \ln\frac{r}{R}. \tag{31}$$

By substituting eqn (31) into eqn (19), after algebraic manipulations, we obtain

$$\mathbf{M}\begin{bmatrix}\bar{K}_{j6}^{u}\\ \bar{K}_{j6}^{v}\\ \bar{K}_{j6}^{w}\end{bmatrix} + \mathbf{M}\begin{bmatrix}\bar{L}_{j6}^{u}\\ \bar{L}_{j6}^{v}\\ \bar{L}_{j6}^{w}\end{bmatrix}\ln\frac{r}{R} + \mathbf{N}\begin{bmatrix}\bar{L}_{j6}^{u}\\ \bar{L}_{j6}^{v}\\ \bar{L}_{j6}^{w}\end{bmatrix} = \mathbf{0} \tag{32}$$

where $\mathbf{M}$ and $\mathbf{N}$ are defined in Table 4. Equation (32) requires that the following equalities be satisfied

$$\mathbf{M}\begin{bmatrix}\bar{L}_{j6}^{u}\\ \bar{L}_{j6}^{v}\\ \bar{L}_{j6}^{w}\end{bmatrix} = \mathbf{0} \quad \text{and} \quad \mathbf{M}\begin{bmatrix}\bar{K}_{j6}^{u}\\ \bar{K}_{j6}^{v}\\ \bar{K}_{j6}^{w}\end{bmatrix} + \mathbf{N}\begin{bmatrix}\bar{L}_{j6}^{u}\\ \bar{L}_{j6}^{v}\\ \bar{L}_{j6}^{w}\end{bmatrix} = \mathbf{0}. \tag{33}$$

The coefficient matrix $\mathbf{M}$ in the first of eqn (33) is singular, and hence the solution of this equation for $[\bar{L}_{j6}^{u} \;\; \bar{L}_{j6}^{v} \;\; \bar{L}_{j6}^{w}]^{T}$ contains one arbitrary parameter. We denote this parameter by G_{j6} and write

$$\bar{L}_{j6}^{u} = 0, \quad \bar{L}_{j6}^{v} = -G_{j6}, \quad \bar{L}_{j6}^{w} = G_{j6}. \tag{34}$$

Table 4. The matrices in eqn (32)

$$\mathbf{M} = \begin{bmatrix} C_{66} & C_{26} & C_{26} \\ C_{26} & (C_{22}+C_{44}) & (C_{22}+C_{44}) \\ C_{26} & (C_{22}+C_{44}) & (C_{22}+C_{44}) \end{bmatrix}$$

$$\mathbf{N} = \begin{bmatrix} 0 & C_{45} & (C_{36}+C_{45}) \\ -C_{45} & 0 & (C_{23}+C_{44}) \\ -(C_{36}+C_{45}) & -(C_{23}+C_{44}) & 0 \end{bmatrix}$$

Substitution of eqns (34) into the second of eqn (33) yields

$$\begin{bmatrix} \bar{K}^u_{j6} \\ \bar{K}^v_{j6} \\ \bar{K}^w_{j6} \end{bmatrix} = -G_{j6} \begin{bmatrix} C_{66} & C_{26} & \\ C_{26} & C_{22}+C_{44} & \\ & & 1 \end{bmatrix}^{-1} \begin{bmatrix} C_{36} \\ C_{23}+C_{44} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ K_{j6} \\ -K_{j6} \end{bmatrix} \tag{35}$$

where (similarly to G_{j6}) K_{j6} is an arbitrary parameter.

The displacements given in eqn (28) and eqn (31) are all appropriate solutions. Hence, if we multiply the displacements in eqn (28) by the constant K_{j6}/G_{j5} and add the resulting displacements to the displacements in eqn (31), we obtain another set of acceptable displacements. The results are

$$\begin{aligned} u^{\text{hom}}_{j6}(r) &= \bar{u}^{\text{hom}}_{j6}(r) + \frac{K_{j6}}{G_{j5}} u^{\text{hom}}_{j5}(r) = K^u_{j6} \\ v^{\text{hom}}_{j6}(r) &= \bar{v}^{\text{hom}}_{j6}(r) + \frac{K_{j6}}{G_{j5}} v^{\text{hom}}_{j5}(r) = K^v_{j6} - G_{j6} \ln \frac{r}{R} \\ w^{\text{hom}}_{j6}(r) &= \bar{w}^{\text{hom}}_{j6}(r) + \frac{K_{j6}}{G_{j5}} w^{\text{hom}}_{j5}(r) = G_{j6} \ln \frac{r}{R} \end{aligned} \tag{36}$$

where

$$\begin{bmatrix} K^u_{j6} \\ K^v_{j6} \end{bmatrix} = -G_{j6} \begin{bmatrix} C_{66} & C_{26} \\ C_{26} & C_{22}+C_{44} \end{bmatrix}^{-1} \begin{bmatrix} C_{36} \\ C_{23}+C_{44} \end{bmatrix}. \tag{37}$$

Equation (36) is the sixth independent solution we have been seeking. The homogeneous solution for $j(\pi/\theta_o) = 1$ thus becomes

$$u^{\text{hom}}_j(r) = \sum_{k=1}^{6} u^{\text{hom}}_{jk}(r), \quad v^{\text{hom}}_j(r) = \sum_{k=1}^{6} v^{\text{hom}}_{jk}(r), \quad w^{\text{hom}}_j(r) = \sum_{k=1}^{6} w^{\text{hom}}_{jk}(r) \tag{38}$$

where for $(k = 1, 2, \ldots, 5)$

$$u^{\text{hom}}_{jk}(r) = G^u_{jk}\left(\frac{r}{R}\right)^{\gamma_k}, \quad v^{\text{hom}}_{jk}(r) = G^v_{jk}\left(\frac{r}{R}\right)^{\gamma_k}, \quad w^{\text{hom}}_{jk}(r) = G^w_{jk}\left(\frac{r}{R}\right)^{\gamma_k} \tag{39}$$

and for $k = 6$

$$u^{\text{hom}}_{j6}(r) = K^u_{j6}, \quad v^{\text{hom}}_{j6}(r) = K^v_{j6} - G_{j6} \ln \frac{r}{R}, \quad w^{\text{hom}}_{j6}(r) = G_{j6} \ln \frac{r}{R}. \tag{40}$$

These equations contain a total of six unknowns G_{jk} $(k = 1, 2, \ldots, 6)$.

Particular solution of the inhomogeneous equation [*eqn* (18)]

A particular solution of the inhomogeneous equation [(eqn 18)] can be written in the form

$$u^{\text{inh}}_j(r) = \sum_{i=0} P^u_{ji}\left(\frac{r}{R}\right)^{i+1}, \quad v^{\text{inh}}_j(r) = \sum_{i=0} P^v_{ji}\left(\frac{r}{R}\right)^{i+1}, \quad w^{\text{inh}}_j(r) = \sum_{i=0} P^w_{ji}\left(\frac{r}{R}\right)^{i+1}. \tag{41}$$

Substitution of eqn (41) into eqn (18) yields

$$\sum_{i=0} F_i \begin{bmatrix} P^u_{ji} \\ P^v_{ji} \\ P^w_{ji} \end{bmatrix} = \sum_{i=0} \Delta T_{ji} R^{i+1} \begin{bmatrix} q_1 j \dfrac{\pi}{\theta_o} \\ q_2 j \dfrac{\pi}{\theta_o} \\ q_2 - (i+1) q_3 \end{bmatrix}. \tag{42}$$

The matrix $\mathbf{F}_i$ is the same as the $\mathbf{E}$ matrix in Table 3, with γ replaced by $(i+1)$. The parameters q_1, q_2, q_3 are given in Table 2. The vector $[P^u_{ji}\ P^v_{ji}\ P^w_{ji}]^T$ is obtained from the equation

$$\begin{bmatrix} P^u_{ji} \\ P^v_{ji} \\ P^w_{ji} \end{bmatrix} = \{\mathbf{F}_i\}^{-1} \Delta T_{ji} R^{i+1} \begin{bmatrix} q_1 j \dfrac{\pi}{\theta_o} \\ q_2 j \dfrac{\pi}{\theta_o} \\ q_2 - (i+1) q_3 \end{bmatrix}. \tag{43}$$

The matrix $\mathbf{F}$ is singular when any one of the roots of eqn (22) $(\gamma_1, \gamma_2, \ldots, \gamma_6)$ is equal to $(i+1)$. This difficulty could be overcome with considerable mathematical and computational complexity. Alternatively, this singularity can be removed without significant loss in accuracy by changing slightly one of the stiffness values in eqn (4).

General solution of the inhomogeneous equation [eqn (18)]

Solution of eqn (18) is the sum of the homogeneous and the inhomogeneous solutions

$$u_j(r) = u_j^{\text{hom}}(r) + u_j^{\text{inh}}(r), \quad v_j(r) = v_j^{\text{hom}}(r) + v_j^{\text{inh}}(r), \quad w_j(r) = w_j^{\text{hom}}(r) + w_j^{\text{inh}}(r). \tag{44}$$

The inhomogeneous displacements are given in eqn (41) and the homogeneous displacements by eqns (38)–(40) for $j(\pi/\theta_o) = 1$ and by eqns (23)–(24) for $j(\pi/\theta_o) \neq 1$.

Complete solution

The analysis presented thus far pertains only to the first part of the series [eqns (15) and (17)]. The second part of this series is

$$-u_j^*(r)\cos j\frac{\pi}{\theta_o}\theta, \quad -v_j^*(r)\cos j\frac{\pi}{\theta_o}\theta, \quad w_j^*(r)\sin j\frac{\pi}{\theta_o}\theta, \quad \left[\sum_{i=0} \Delta T_{ji}^*(r)^i\right]\sin j\frac{\pi}{\theta_o}\theta. \tag{45}$$

The solution for these terms proceeds along the same line as for the "unstarred" terms in eqns (18)–(44). We merely need to replace in eqns (18)–(44) the "unstarred" constants $(G, L, \bar{L}, \bar{K})$ with "starred" constants $(G^*, L^*, \bar{L}^*, \bar{K}^*)$.

6. RADIALLY AND CIRCUMFERENTIALLY VARYING STRAINS AND STRESSES (CURVED AXIS)

We consider the problem of a cylinder whose axis has curvatures in the x–y and x–z planes (κ^y, κ^z). We now write the displacements in the form

$$u_B(x,\theta,r) = u_B^y(x,\theta,r) + u_B^z(x,\theta,r), \quad v_B(x,\theta,r) = v_B^y(x,\theta,r) + v_B^z(x,\theta,r),$$
$$w_B(x,\theta,r) = w_B^y(x,\theta,r) + w_B^z(x,\theta,r). \tag{46}$$

The first displacements on the right-hand side (superscript y) are the displacements due to curvature in the x–y plane, and the second terms (superscript z) are the displacements due to the curvature in the x–z plane. First, we derive expressions for u_B^y, v_B^y, w_B^y. These displacements are written as

$$u_B^y(x,\theta,r) = \kappa^y xr\cos\theta + u_H^y(r)\sin\theta,\quad v_B^y(x,\theta,r) = \kappa^y\frac{x^2}{2}\sin\theta + v_H^y(r)\sin\theta,$$

$$w_B^y(x,\theta,r) = -\kappa^y\frac{x^2}{2}\cos\theta + w_H^y(r)\cos\theta. \tag{47}$$

These displacements together with the strain–displacement eqn (3), stress–strain eqn (4) and the equilibrium equation (2) yield

$$\begin{bmatrix}\Lambda_{11} & \Lambda_{12} & \Lambda_{13}\\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23}\\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33}\end{bmatrix}\begin{bmatrix}u_H^y(r)\\ v_H^y(r)\\ w_H^y(r)\end{bmatrix} = \kappa^y\begin{bmatrix}-C_{61}\\ -C_{21}\\ 2C_{31}-C_{21}\end{bmatrix}. \tag{48}$$

Λ_{ij} is the same as Ω_{ij} in Table 2 with $j(\pi/\theta_o)$ set equal to 1.

By comparing eqns (48) and (18) we observe that the homogeneous form of these equations (i.e. the right-hand sides set equal to zero) are similar. Thus, by referring to eqns (38)–(40) we can write the solution for u_H^y, v_H^y, w_H^y as

$$u_H^{y\,\mathrm{hom}}(r) = \sum_{k=1}^{6} u_{Hk}^{y\,\mathrm{hom}}(r),\quad v_H^{y\,\mathrm{hom}}(r) = \sum_{k=1}^{6} v_{Hk}^{y\,\mathrm{hom}}(r),\quad w_H^{y\,\mathrm{hom}}(r) = \sum_{k=1}^{6} w_{Hk}^{y\,\mathrm{hom}}(r) \tag{49}$$

where for ($k = 1, 2, \ldots, 5$)

$$u_{Hk}^{y\,\mathrm{hom}}(r) = H_k^{yu}\left(\frac{r}{R}\right)^{\gamma_k},\quad v_{Hk}^{y\,\mathrm{hom}}(r) = H_k^{yv}\left(\frac{r}{R}\right)^{\gamma_k},\quad w_{Hk}^{y\,\mathrm{hom}}(r) = H_k^{yw}\left(\frac{r}{R}\right)^{\gamma_k} \tag{50}$$

and for $k = 6$

$$u_{H6}^{y\,\mathrm{hom}}(r) = J_6^{yu},\quad v_{H6}^{y\,\mathrm{hom}}(r) = J_6^{yv} - H_6^y\ln\frac{r}{R},\quad w_{H6}^{y\,\mathrm{hom}}(r) = H_6^y\ln\frac{r}{R}. \tag{51}$$

Note the similarity with eqns (39) and (40); H and J correspond to G and K, the only difference being that H and J are now evaluated by eqns (25), (26) and (37) with $j(\pi/\theta_o) = 1$.

The exponents γ_k are the five independent roots of eqn (22) with $j(\pi/\theta_o) = 1$ and with $\gamma_5 = 0$. The above homogeneous solutions [eqns (49)–(51)] contain six independent constants H_k^y ($k = 1, 2, \ldots, 6$), where H_5^y represents rigid body motion in the $\theta = 0$ direction.

A particular solution of the inhomogeneous equation [eqn (48)] is

$$u_H^{y\,\mathrm{inh}}(r) = S^{yu}\left(\frac{r}{R}\right)^2,\quad v_H^{y\,\mathrm{inh}}(r) = S^{yv}\left(\frac{r}{R}\right)^2,\quad w_H^{y\,\mathrm{inh}}(r) = S^{yw}\left(\frac{r}{R}\right)^2. \tag{52}$$

Substitution of eqn (52) into eqn (48) yields

$$\begin{bmatrix}S^{yu}\\ S^{yv}\\ S^{yw}\end{bmatrix} = \kappa^y R^2\begin{bmatrix}C_{66}-4C_{55} & C_{26}-2C_{45} & C_{26}+2(C_{36}+C_{45})\\ C_{26}-6C_{45} & -3C_{44}+C_{22} & C_{22}+3C_{44}+2C_{23}\\ C_{26}-2(C_{36}+C_{45}) & C_{22}-C_{44}-2C_{23} & C_{22}+C_{44}-4C_{33}\end{bmatrix}^{-1} * \begin{bmatrix}-C_{61}\\ -C_{21}\\ 2C_{31}-C_{21}\end{bmatrix}. \tag{53}$$

The displacements caused by curvature κ^y are [eqns (47), (49) and (52)]

$$u_B^y(x,\theta,r) = \kappa^y xr\cos\theta + u_H^y(r)\sin\theta = \kappa^y xr\cos\theta + \left(u_H^{y\,hom}(r) + u_H^{y\,inh}(r)\right)\sin\theta$$

$$v_B^y(x,\theta,r) = \kappa^y\frac{x^2}{2}\sin\theta + v_H^y(r)\sin\theta = \kappa^y\frac{x^2}{2}\sin\theta + \left(v_H^{y\,hom}(r) + v_H^{y\,inh}(r)\right)\sin\theta$$

$$w_B^y(x,\theta,r) = -\kappa^y\frac{x^2}{2}\cos\theta + w_H^y(r)\cos\theta = -\kappa^y\frac{x^2}{2}\cos\theta + \left(w_H^{y\,hom}(r) + w_H^{y\,inh}(r)\right)\cos\theta. \quad (54)$$

The displacements caused by curvature κ^z can be derived in a similar manner. The result is

$$u_B^z(x,\theta,r) = \kappa^z xr\sin\theta - u_H^z(r)\cos\theta = \kappa^z xr\sin\theta - \left(u_H^{z\,hom}(r) + u_H^{z\,inh}(r)\right)\cos\theta$$

$$v_B^z(x,\theta,r) = -\kappa^z\frac{x^2}{2}\cos\theta - v_H^z(r)\cos\theta = -\kappa^z\frac{x^2}{2}\cos\theta - \left(v_H^{z\,hom}(r) + v_H^{z\,inh}(r)\right)\cos\theta$$

$$w_B^z(x,\theta,r) = -\kappa^z\frac{x^2}{2}\sin\theta + w_H^z(r)\sin\theta = -\kappa^z\frac{x^2}{2}\sin\theta + \left(w_H^{z\,hom}(r) + w_H^{z\,inh}(r)\right)\sin\theta. \quad (55)$$

The homogeneous and particular solutions are the same as given before by eqns (49) and (52) for the y component. The differences are that H_k^{yu}, H_k^{yv}, H_k^{yw}, H_k^{y}, J_6^{yu}, J_6^{yv} and κ^y are replaced by H_k^{zu}, H_k^{zv}, H_k^{zw}, H_k^{z}, J_6^{zu}, J_6^{zv} and κ^z.

7. STRAINS AND STRESSES

Using the expressions for the displacements derived in the foregoing sections, the strains can be calculated from eqn (3) and the stresses from eqn (4). In the analysis of cylinders and cylindrical segments we will make use of the stresses obtained in this manner. Therefore, the stresses are tabulated in Table 5. The results in this table show that the dependence of the stresses on r and θ are separated. The stress components with a "hat" depend only on the radius r.

8. NUMBER OF UNKNOWN CONSTANTS

The expressions for displacements contain a number of unknown constants, as summarized in Table 6. These constants must be determined by applying continuity conditions across ply interfaces, conditions for no rigid body motion, and appropriate boundary conditions.

9. CONTINUITY CONDITIONS

At each ply interface the displacements and three of the stresses $(\sigma_r, \tau_{r\theta}, \tau_{rx})$ must be the same in adjacent layers. Thus, at the interface between the l and $l+1$ layers (Fig. 7) the continuity conditions given in Tables 7–10 must be satisfied.

For u_o, v_o and w_o the equations in Table 8 represent $(n-1)*10$ equations for a composite made of n layers. Each layer contains 10 unknowns, so the total number of unknowns is $10*n$. From the above set of equations all but 10 of these unknowns can be determined.

For u_F, v_F and w_F the equations in Table 9 represent $(n-1)*6*2$ equations for an n ply composite for every Fourier term. In each layer there are 6*2 unknowns for each term. Of these $(n-1)*6*2$ can be determined from these equations. There remain 6*2 unknowns for each term.

Table 5. The displacement, temperature and stress terms

	u_0, v_0, w_0	u_F, v_F, w_F		u_B, v_B, w_B	
u		$u_j(r)\sin j\frac{\pi}{\theta_0}\theta$	$-u_j^*(r)\cos j\frac{\pi}{\theta_0}\theta$	u_B^y, v_B^y, w_B^y	u_B^z, v_B^z, w_B^z
v		$v_j(r)\sin j\frac{\pi}{\theta_0}\theta$	$-v_j^*(r)\cos j\frac{\pi}{\theta_0}\theta$		
w		$w_j(r)\cos j\frac{\pi}{\theta_0}\theta$	$w_j^*(r)\sin j\frac{\pi}{\theta_0}\theta$		
ΔT	$\sum_{i=0}\Delta T_{0i}r^i$	$\left[\sum_{i=0}\Delta T_{ji}(r)^i\right]\cos j\frac{\pi}{\theta_0}\theta$	$\left[\sum_{i=0}\Delta T_{ji}^*(r)^i\right]\sin j\frac{\pi}{\theta_0}\theta$		
σ_x	σ_{x0}	$\sigma_{xj} = \hat{\sigma}_{xj}\cos j\frac{\pi}{\theta_0}\theta$	$\sigma_{xj}^* = \hat{\sigma}_{xj}^*\sin j\frac{\pi}{\theta_0}\theta$	$\sigma_{xB}^y = \hat{\sigma}_{xB}^y\cos\theta$	$\sigma_{xB}^z = \hat{\sigma}_{rB}^z\sin\theta$
σ_θ	$\sigma_{\theta 0}$	$\sigma_{\theta j} = \hat{\sigma}_{\theta j}\cos j\frac{\pi}{\theta_0}\theta$	$\sigma_{\theta j}^* = \hat{\sigma}_{\theta j}^*\sin j\frac{\pi}{\theta_0}\theta$	$\sigma_{\theta B}^y = \hat{\sigma}_{\theta B}^y\cos\theta$	$\sigma_{\theta B}^z = \hat{\sigma}_{\theta B}^z\sin\theta$
σ_r	σ_{r0}	$\sigma_{rj} = \hat{\sigma}_{rj}\cos j\frac{\pi}{\theta_0}\theta$	$\sigma_{rj}^* = \hat{\sigma}_{rj}^*\sin j\frac{\pi}{\theta_0}\theta$	$\sigma_{rB}^y = \hat{\sigma}_{rB}^y\cos\theta$	$\sigma_{rB}^z = \hat{\sigma}_{rB}^z\sin\theta$
$\tau_{r\theta}$	$\tau_{r\theta 0}$	$\tau_{r\theta j} = \hat{\tau}_{r\theta j}\sin j\frac{\pi}{\theta_0}\theta$	$\tau_{r\theta j}^* = \hat{\tau}_{r\theta j}^*\cos j\frac{\pi}{\theta_0}\theta$	$\tau_{r\theta B}^y = \hat{\tau}_{r\theta B}^y\sin\theta$	$\tau_{r\theta B}^z = \hat{\tau}_{r\theta B}^z\cos\theta$
τ_{rx}	τ_{rx0}	$\tau_{rxj} = \hat{\tau}_{rxj}\sin j\frac{\pi}{\theta_0}\theta$	$\tau_{rxj}^* = \hat{\tau}_{rxj}^*\cos j\frac{\pi}{\theta_0}\theta$	$\tau_{rxB}^y = \hat{\tau}_{rxB}^y\sin\theta$	$\tau_{rxB}^z = \hat{\tau}_{rxB}^z\cos\theta$
$\tau_{x\theta}$	$\tau_{x\theta 0}$	$\tau_{x\theta j} = \hat{\tau}_{x\theta j}\cos j\frac{\pi}{\theta_0}\theta$	$\tau_{x\theta j}^* = \hat{\tau}_{x\theta j}^*\sin j\frac{\pi}{\theta_0}\theta$	$\tau_{x\theta B}^y = \hat{\tau}_{x\theta B}^y\cos\theta$	$\tau_{x\theta B}^z = \hat{\tau}_{x\theta B}^z\sin\theta$

Table 6. The unknown constants in the displacements, continuity, no rigid body motion and boundary conditions

	u_0^l, v_0^l, w_0^l	u_F^l, v_F^l, w_F^l		u_B^l, v_B^l, w_B^l
		$j\frac{\pi}{\theta_0} \neq 1$	$j\frac{\pi}{\theta_0} = 1$	
Unknowns (one layer)	A_1^l, A_2^l $u_a^l, u_b^l, u_c^l, u_d^l$ $v_a^l, v_b^l, v_c^l, v_d^l$	G_{jk}^l $k = 1, 2, \ldots, 6$ G_{jk}^{l*} $j = 1, 2, \ldots,$ number of Fourier terms		κ^y, H_k^{ly} $k = 1, 2, \ldots, 6$ κ^z, H_k^{lz}
Number of unknowns	$10 * n$	$(2 * 6)n$ for each Fourier term		$2 + (2 * 6)n$
Continuity conditions	$10 * (n - 1)$	$(2 * 6)(n - 1)$ for each Fourier term		$(2 * 6)(n - 1)$
No rigid body motion	$u_d^1 = 0,\ v_d^1 = 0$		$G_{j5}^1 = 0,\ G_{j5}^{1*} = 0$	$H_5^{1y} = 0,\ H_5^{1z} = 0$
Boundary conditions if there are no rigid body motions	8	$2 * 6$ for each Fourier term	$2 * 5$	$2 * 6$

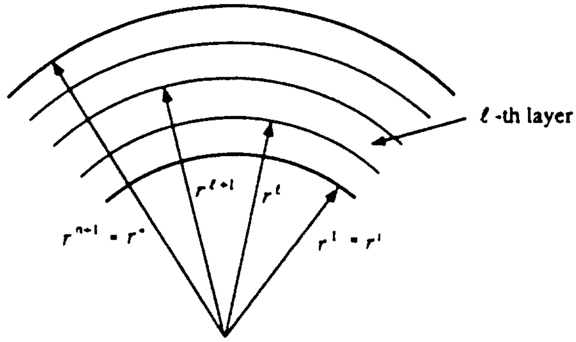

Fig. 7. Numbering of the plies.

Table 7. The continuity conditions at the interface between the l and $(l+1)$ layers

$$u^l(x,\theta,r) = u^{l+1}(x,\theta,r)$$
$$v^l(x,\theta,r) = v^{l+1}(x,\theta,r) \quad (r = r^{l+1}) \quad (l = 1,2,\ldots,n-1)$$
$$w^l(x,\theta,r) = w^{l+1}(x,\theta,r)$$

$$\sigma_r^l(x,\theta,r) = \sigma_r^{l+1}(x,\theta,r)$$
$$\tau_{r\theta}^l(x,\theta,r) = \tau_{r\theta}^{l+1}(x,\theta,r) \quad (r = r^{l+1}) \quad (l = 1,2,\ldots,n-1)$$
$$\tau_{rx}^l(x,\theta,r) = \tau_{rx}^{l+1}(x,\theta,r)$$

Table 8. The continuity conditions for u_o, v_o, w_o

The displacement continuity conditions

$$u_a^l = u_a^{l+1}$$
$$u_b^l = u_b^{l+1}$$
$$u_c^l(r) = u_c^{l+1}(r) \quad (r = r^{l+1})$$
$$v_a^l = v_a^{l+1}$$
$$v_b^l = v_b^{l+1}$$
$$v_c^l(r) = v_c^{l+1}(r) \quad (r = r^{l+1})$$
$$w_o^l(r) = w_o^{l+1}(r) \quad (r = r^{l+1})$$

The stress continuity conditions

$$\sigma_{ro}^l(r) = \sigma_{ro}^{l+1}(r)$$
$$\tau_{r\theta o}^l(r) = \tau_{r\theta o}^{l+1}(r) \quad (r = r^{l+1})$$
$$\tau_{rxo}^l(r) = \tau_{rxo}^{l+1}(r)$$

Table 9. The continuity conditions for u_I, v_I, w_I

The displacement continuity conditions

$$u_j^l(r) = u_j^{l+1}(r) \quad u_j^{l*}(r) = u_j^{l+1*}(r)$$
$$v_j^l(r) = v_j^{l+1}(r) \quad v_j^{l*}(r) = v_j^{l+1*}(r) \quad (r = r^{l+1})$$
$$w_j^l(r) = w_j^{l+1}(r) \quad w_j^{l*}(r) = w_j^{l+1*}(r)$$

The stress continuity conditions

$$\hat{\sigma}_{rj}^l(r) = \hat{\sigma}_{rj}^{l+1}(r) \quad \hat{\sigma}_{rj}^{l*}(r) = \hat{\sigma}_{rj}^{l+1*}(r)$$
$$\hat{\tau}_{r\theta j}^l(r) = \hat{\tau}_{r\theta j}^{l+1}(r) \quad \hat{\tau}_{r\theta j}^{l*}(r) = \hat{\tau}_{r\theta j}^{l+1*}(r) \quad (r = r^{l+1})$$
$$\hat{\tau}_{rxj}^l(r) = \hat{\tau}_{rxj}^{l+1}(r) \quad \hat{\tau}_{rxj}^{l*}(r) = \hat{\tau}_{rxj}^{l+1*}(r)$$

Table 10. The continuity conditions for u_{II}, v_{II}, w_{II}

The displacement continuity conditions

$$u_{II}^{lz}(r) = u_{II}^{(l+1)z}(r) \quad u_{II}^{ly}(r) = u_{II}^{(l+1)y}(r)$$
$$v_{II}^{lz}(r) = v_{II}^{(l+1)z}(r) \quad v_{II}^{ly}(r) = v_{II}^{(l+1)y}(r) \quad (r = r^{l+1})$$
$$w_{II}^{lz}(r) = w_{II}^{(l+1)z}(r) \quad w_{II}^{ly}(r) = w_{II}^{(l+1)y}(r)$$

The stress continuity conditions

$$\hat{\sigma}_{rII}^{lz}(r) = \hat{\sigma}_{rII}^{(l+1)z}(r) \quad \hat{\sigma}_{rII}^{ly}(r) = \hat{\sigma}_{rII}^{(l+1)y}(r)$$
$$\hat{\tau}_{r\theta II}^{lz}(r) = \hat{\tau}_{r\theta II}^{(l+1)z}(r) \quad \hat{\tau}_{r\theta II}^{ly}(r) = \hat{\tau}_{r\theta II}^{(l+1)y}(r) \quad (r = r^{l+1})$$
$$\hat{\tau}_{rxII}^{lz}(r) = \hat{\tau}_{rxII}^{(l+1)z}(r) \quad \hat{\tau}_{rxII}^{ly}(r) = \hat{\tau}_{rxII}^{(l+1)y}(r)$$

For u_B, v_B and w_B the equations in Table 10 represent $(n-1)*6*2$ equations for a composite made of n layers. Each layer contains 6*2 unknowns. Furthermore κ^y and κ^z are also unknowns. Hence the total number of unknowns is $6*2*n+2$. From the above set of equations all but 14 of these unknowns can be determined.

10. RIGID BODY MOTION

As was discussed above (eqns 11, 12, 28 and 50), rigid body motion is represented by the constants u_d^l, v_d^l, G_{j5}^l, G_{j5}^{l*}, H_5^{ly}, H_5^{lz}. In the absence of rigid body motion these constants must be zero in one of the plies. For convenience, we prescribe these constants for the innermost ply. Thus, for the displacements u_o, v_o, w_o we have

$$u_d^1 = 0, \quad v_d^1 = 0. \tag{56}$$

For u_F, v_F, w_F, $(j(\pi/\theta_o) = 1)$ and for u_B, v_B, w_B the conditions for no rigid body motion are

$$G_{j5}^1 = 0, \quad G_{j5}^{1*} = 0 \tag{57}$$

$$H_5^{1y} = 0, \quad H_5^{1z} = 0. \tag{58}$$

Equations (56)–(58) eliminate $3*2 = 6$ constants. The remaining constants must be found with the aid of the continuity and boundary conditions.

11. BOUNDARY CONDITIONS

The conditions for rigid body motions and the continuity conditions provide some but not all the equations needed to determine all the unknown constants in Table 6. The additional equations required to determine all the constants are provided by the boundary conditions. Appropriate boundary conditions for closed cylinders are presented in a companion paper (Kollár *et al.*, 1992). Boundary conditions for cylindrical segments, and flat panels joined by curved corners will be described in subsequent publications.

Acknowledgements—This work was supported by an Imre Korányi fellowship provided to LPK through the Thomas Cholnoky Foundation. This support is gratefully acknowledged.

REFERENCES

Chandrashekhara, K. and Gopalakrishnan, P. (1982). Elasticity solution for a multilayered transversely isotropic circular cylindrical shell. *J. Appl. Mech.* **49**, 108–114.

Chou, F. H. and Achenbach, J. D. (1972). Three dimensional vibration of orthotropic cylinders. *J. Engng Mech. ASCE* **98**(EM4), 813–822.

Grigorenko, Ya. M., Vasilenko, A. T. and Pankratova, N. D. (1974). Computation of the stressed state of thick-walled inhomogeneous anisotropic shells. *Prikladadnaya Mekhanika* **10**, 86–93 (in Russian); (English translation in *Soviet Appl. Mech.* **10**, 523–528).

Hyer, M. W. (1988). Hydrostatic response of thick laminated composite cylinders. *J. Reinforced Plastic and Composites* **7**, 321–340.

Hyer, M. W., Cooper, D. E. and Cohen, D. (1986). Stresses and deformations in cross-ply composite tubes subjected to uniform temperature change. *J. Thermal Stresses* **9**, 97–117.

Kollár, L. P., Patterson, J. M. and Springer, G. S. (1992). Composite cylinders subjected to hygrothermal and mechanical loads. *Int. J. Solids Structures* **29**, 1519–1534.

Lee, S. Y. and Springer, G. S. (1990). Filament winding cylinders: I. Process model. *J. Comp. Mater.* **24**, 1270–1298.

Love, A. E. H. (1944). *A Treatise on the Mathematical Theory of Elasticity* (4th Edn). Dover, New York.

Noor, A. K., Burton, W. S. and Peters, J. M. (1991). Assessment of computational models for multilayered composite cylinders. *Int. J. Solids Structures* **27**, 1269–1286.

Noor, A. K. and Peters, J. M. (1989). Stress, vibration and buckling of multilayered cylinders. *J. Struct. Engng* **115**, 69–88.

Noor, A. K. and Rarig, P. L. (1974). Three dimensional solutions of laminated cylinders. *Comput. Meth. Appl. Mech. Engng* **3**, 319–334.

Ren, J. G. (1987). Exact solutions for laminated cylindrical shells in cylindrical bending. *Comp. Sci. Tech.* **29**, 169–187.

Roy, A. K. and Tsai, S. W. (1988). Design of thick composite cylinders. *J. Pressure Vessel Tech. ASME* **110**, 255–262.

Spencer, A. J. M., Watson, P. and Rogers, T. G. (1990). Stress analysis of anisotropic laminated circular cylindrical shells. *ASME*, AMD-Vol. 113, 57–60.

Srinivas, S. (1974). Analysis of laminated, composite, circular cylindrical shells with general boundary conditions. NASA TR-R-412.

Tsai, S. W. (1988). Composites Design (4th Edn). Think Composites, Dayton.

Varadan, T. K. and Bhaskar, K. (1991). Bending of laminated orthotropic cylindrical shells—an elastic approach. *Comp. Struct.* **17**, 141–156.

Wolfram, S. (1988). *Mathematica. A System for Doing Mathematics by Computer*. Addison-Wesley, Redwood City.